**Activity 3.4.3 Pascal’s Triangle and the Binomial Theorem**

The terms of the expansion of *(a+b)n* are often referred to as the Binomial Theorem. This activity will show how to use Pascal’s Triangle to find the coefficients of each term in the expansion. Continue to add rows to Pascal’s Triangle below as needed to answer the following problems.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

We will examine the powers of the binomial *(a+b)n* by finding the products for n = 0, 1, 2, 3, 4, and 5. The expansion of the binomial for n = 0 and n = 1 are given. Remember that the coefficient equals 1 when there is no coefficient present. The 1’s are inserted in (a+b)1 to illustrate this idea.

*(a+b)0 = 1*

*(a+b)1 = 1•a + 1•b*

*(a+b)2 =*

*(a+b)3 =*

*(a+b)4 =*

*(a+b)5 =*

1. Based on the pattern you see in your answers, predict what the expansion of (a+b)6 would equal.

2. Expand *(a+b)10* using Pascal’s Triangle. Check your answer with a CAS.

3. Describe in general how you can find the coefficients of the terms in the expansion of *(a+b)n*.

The Binomial Theorem states the expression that is obtained when the binomial *(a+b)* is raised to the nth power for all positive integers, n. For the purpose of this class, we will use the notation *cn,i* to represent the coefficient of the term in the binomial expansion in which *b* is raised to the *ith* power. Each *cn,i* corresponds to the *ith* entry in the n+1st row of Pascal’s Triangle and for *i=0,1,2,…,n.* Using this notation, we can state the Binomial Theorem as follows:

For all positive integers, *n = 1, 2, •••,* the *nth* power of the binomial *a+b*  is given by:

*(a+b)n = cn,o•an•b0 + cn,1•an–1•b1 + cn,2•an–2•b2 +. . . . . . +cn,n–1•a1•bn–1 + cn,n•a0•bn*

Therefore, the Binomial Theorem states an identity that can be used to expand other binomials raised to a power.

Example 1:

Expand (x+1)3 using the Binomial Theorem. Replacing x for a and 1 for b, we obtain the expansion using the 4th row in Pascal’s Triangle.

The row to use in the expansion is: 1 3 3 1

Therefore:

(x+1)3 = 1•x3 + 3•x2•1 + 3•x•12 + 1•13 or

(x+1)3 = x3 + 3x2 + 3x + 1

Example 2:

Expand (2x – 3y2)4. Replacing 2x for a and –3y2 for b, we obtain the expansion using the 5th row in Pascal’s Triangle.

Therefore:

(2x – 3y2)4 = 1•(2x)4 + 4•(2x)3(–3y2) + 6•(2x)2(–3y2)2 + 4•(2x)(–3y2)3 + 1•(–3y2)4

= 16x4 + 4•8x3•(–3y2) + 6•4x2•9y4 + 4•2x•(–27y6) + 81y8

= 16x4 – 96x3y2 + 216x2y4 – 216xy6 + 81y8

1. Exercises to Practice:
   1. (x+5)4
   2. (4y – 3x)5
   3. 
   4. ( ½ + ½ )6
2. The Binomial Theorem can also be applied to numerical computations. Fill in the table below to see how to expand the powers of 11.

|  |  |
| --- | --- |
| 110 = (10+1)0 | 1 |
| 111= (10+1)1 | 11 |
| 112= (10+1)2 | 121 |
| 113= (10+1)3 | 1331 |
| 114= (10+1)4 |  |
| 115= (10+1)5 |  |
| 116= (10+1)6 |  |

Try to determine how you can use the coefficients of the binomial theorem to expand 11n as (10+1)n.

Use your conjecture to find 1110.

1. Application to Probability

The probability that a child will be born with blue eyes when both the mother and the father have brown eyes, but are carriers of the gene for blue eyes, is ¼. Thus the probability that the child will be born with brown eyes is ¾. If the terms of the expansion of (¼ + ¾)4 give the probabilities that 4, 3, 2, 1, and 0 children will have blue eyes if the couple has 4 children, what is the probability that two children will have blue eyes and two children will have brown eyes?