**Unit 3: Investigation 4 (+) (3 days)**

**Binomial Theorem and Its Applications**

**Common Core State Standards**

A.APR.5 *(+)* Know and apply that the Binomial Theorem gives the expansion of (x + y)n in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal’s Triangle. (The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.)

**Overview**

Investigation 4 examines the application of multiplication of polynomials to the Binomial Theorem which expresses how to find the product (x+y)n. The investigation will introduce the concept using a combinatorial approach by studying the number of subsets of a set with n elements. In studying the pattern of the number of possible 0-element, 1-element, 2-element, ••• n-element subsets of a set of n elements with varying values of n, the students will construct a model of Pascal’s Triangle. Once developed, the students will then study the expansion of the powers of the binomial (a+b). This leads to the generalization of the expansion of (a+b)n and the binomial theorem. Applications of the Binomial Theorem can then find the power of any expression given in the form of a binomial and solve problems involving combinations of a number of objects. Please note this is a + investigation.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Construct the array of numbers known as Pascal’s Triangle and know that the entries in the nth line in the array represent the number of subsets containing 0, 1, 2, . . . . , n–1 elements from left to right.
* Apply the Binomial Theorem to expand any power of a binomial expression of the form (x + y)n.
* Apply the binomial expansion to compute the number of combinations of n objects taken r at a time.

**Assessment Strategies: How Will They Show What They Know?**

* **Exit Slip 3.4** asks students to work in pairs to create their own binomials to expand and then have their partner perform the expansion using the Binomial Theorem. Afterwards each student will check the work of their partner.
* **Journal Prompt 1** asks students to try to explain why so many pizzas can be made with so few toppings.
* **Journal Prompt 2** asks students to describe the symmetry in Pascal’s Triangle and develop an explanation for the symmetry.
* **Activity 3.4.1** **False Advertising: How Many Pizzas Can They Make?** has students explore the number of pizza types one can have if 5 toppings are available. Then the problem is expanded to n available toppings.
* **Activity 3.4.2 Pascal’s Pizza Parlor!** requires thatstudents generalize the pattern in Pascal’s Triangle in order to find the next line in the triangle and the sum of each line in the triangle.
* **Activity 3.4.3** **Pascal’s Triangle and the Binomial Theorem** establishes the link between Pascal’s Triangle and the Binomial Theorem, which in turn allows students to expand expressions of the form *(a + b)n,* where students replace *a* and *b* by the term in the binomial and expand the expression.

**Launch Notes**

The number of subsets of a set of n objects is introduced by having students investigate how many different pizzas they could create with a set of only 5 possible toppings to put on the pizza including the plain cheese pizza. After students resolve this problem, they will further investigate what happens when the problem is extended to n possible toppings. By studying the pattern of what happens for n = 0, 1, 2, ••• possible toppings, students will note that the pattern of possible subsets of the n toppings forms what is known as Pascal’s Triangle. From there, the development of the Binomial Theorem will follow and be applied to expand any binomial of the form (x + y)n.

**Teaching Strategies**

**Activity 3.4.1** will launch the investigation with the following problem. A local pizzeria advertises that they can make more than 30 different pizzas. However, upon arriving at the pizzeria, you see that they only offer 5 different toppings besides cheese that can be put on a pizza. Can the owner be accused of false advertising? The activity is designed to be open-ended in order to allow students multiple ways of solving the problem. Most students will decide to list the possible options, sometimes using a systematic approach, other times not. When reviewing the answers from the class, it should be noted that the answer represents all the possible subsets of the set of 5 toppings, and that the plain cheese pizza, which is one of the options, corresponds to the empty set, also a subset of the original set of toppings. Students should make as many observations about the answer as possible, but the follow-up activity will examine questions regarding the total number of subsets and the symmetry between the entries in Pascal’s Triangle.

**Journal Prompt 1:** Ask the students why they think so many pizzas can be made with so few toppings. Students should note that because there are so many combinations of the five toppings, a lot of different pizzas can be made.

**Group Activity:**

Place the students in pairs of approximately equal ability to complete Activity 3.4.1 computing the number of different pizzas that can be made with 5 toppings. While this is a counting problem, it leads to the creation of Pascal’s Triangle, the base from which the Binomial Theorem will emerge.

**Differentiated Instruction:**

Depending on the level of the student, hints and scaffolding can be given. For example, students can be asked to find how many different pizzas can be made with 0, 1, 2, 3, 4, or 5 toppings. Another way to scaffold the problem is to develop a pattern for the number of pizzas you could make starting with 0 toppings to choose from; 1 topping to choose from; 2 toppings to choose from; and continue up to 5 toppings. Students would then have the beginning of Activity 3.4.2 that will determine the values of Pascal’s Triangle.

In **Activity 3.4.2**, students will extend their results from Activity 3.4.1 and develop the values of Pascal’s Triangle by looking at the pattern formulated by the number of pizzas formed with the different number of toppings to choose from and the number of toppings chosen. The table below could be used to guide the students in developing the pattern.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| No. of Toppings | Pizzas with 0 toppings | Pizzas with 1 toppings | Pizzas with 2 toppings | Pizzas with 3 toppings | Pizzas with 4 toppings | Pizzas with 5 toppings | Pizzas with 6 toppings | Total # of toppings |
| 0 | 1 |  |  |  |  |  |  | 1 |
| 1 | 1 | 1 |  |  |  |  |  | 2 |
| 2 |  |  |  |  |  |  |  |  |
| 3 |  |  |  |  |  |  |  |  |
| 4 |  |  |  |  |  |  |  |  |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 |  | 32 |
| 6 |  |  |  |  |  |  |  |  |

Questions that students could answer include: What patterns to you see in the table? Is there a way to predict what the numbers would be if there were 6 toppings to choose from without figuring out all the individual possibilities? Is there a pattern to the total number of pizzas? What symmetry can you find in the table? Can you explain why the symmetry exists?

The combinatorics argument for the pattern in Pascal’s Triangle is that the number of r-element subsets plus the number of r+1-element subsets of a set of n elements will equal the number of r+1-element subsets of a set of n+1 elements. The rationale for this is that when you add an extra element to the set with n elements, you can add the extra element from the n+1 elements to all the r-element subsets of the first n elements to form subsets with r+1 elements. Then, to that answer, add the number of r+1-element subsets of the first n elements. This will yield all the r+1-element subsets of the n+1 elements.

**Journal Prompt 2:** Give the students the following statement: “Notice that there is symmetry in Pascal’s Triangle. Describe the symmetry in the array and try to explain reasons behind the symmetry.” Students should notice that the numbers in the triangle go up and then down in the same pattern from left to right. At this point, a good question to ask is why the number of 1 topping pizzas would equal the number of pizzas with one less that all the toppings. Some students may see that there is a 1-1 correspondence between these sets.

**Group Activity:**

Students will work in groups on the activity and report their generalizations to the class.

**Differentiated Instruction:**

Students in upper level classes may not require the scaffolding afforded in the table above. Such students may be able to create the table on their own and see the pattern in the numbers.

In **Activity 3.4.3**, students will make the connection between the coefficients of the expansion of the binomial (x+1)n and the numbers in Pascal’s Triangle. Students will create a table with the terms of the expansion of (x+1)1; (x+1)2; (x+1)3; • • • ; (x+1)n.

The explanation for the connection between the coefficient of xr in the binomial expansion and the (n–r+1)th entry in the n+1th row can be seen as counting how many ways you can get r factors of x from the n factors of (x+1).

Once the expansion of the binomial is linked to Pascal’s triangle, students can then apply their results to the expansion of binomials like (x+y)n; (2a + 3b)5; (½+ ½)6; and other problems in which the expansion of a binomial is an appropriate model. Two additional problems involve applications of the Binomial Theorem. The first problem, intended for more advanced students, examines the powers of 11, written as (10+1)n. An initial pattern in the powers of 11 reveal the entries of Pascal’s Triangle, but when you arrive at (10+1)5, students must examine what happens with the entries of Pascal’s Triangle to obtain the solution. Teacher notes for this portion of the activity are found at the end of the Activity 3.4.3 Answer Key as well as they are printed below in the resources. Additional explanation is included below in the Differentiated Instruction box. A second problem involves genetics and computes the probabilities of how many children will have blue eyes or brown eyes in a family of 4 children when the probability of a child having blue eyes when both parents are carriers of the gene for blue eyes is ¼. The sample space for this probability experiment is modeled by the expansion of the binomial ( ¼B + ¾b )4, where B represents a child with blue eyes and b, a child with brown eyes.

**Exit Slip 3.4** has the students work in pairs to create and check binomial expansion examples.

**Group Activity:**

Students will work in groups to find the expanded form of the powers of (x+y)n and look for patterns in the coefficients of the expansions. They will make and report their generalizations to the class.

**Differentiated Instruction:**

The level of examples when applying the Binomial Theorem to expand binomials in general may be adapted to the level of the students.

As a differentiated enrichment activity or as an alternative performance task, students can examine the powers of 11 and see how the Binomial Theorem can be applied to the expansion of (10 + 1)n and how the entries in Pascal’s Triangle become the number of 1’s, 10’s, •••, 10n’s in the expansion. It becomes a good review task to trade in the excess in each place value to the next place values to produce the power of 11. For example, 116 can be determined by the entries in the row of Pascal’s triangle 1 6 15 20 15 6 1. Working from right to left, i.e., starting in the 1’s column, we see that:

• there is 1 in the 1’s column;

• 6-10’s, so 6 goes in the 10’s column;

• 15-100’s or 1-1,000 and 5-100’s, so a 5 goes in the 100’s column;

• which in turn gives 21-1,000’s or 2-10,000’s and 1-1,000, so a 1 goes in the 1000’s column.

Continuing this process, we obtain 17-10,000’s or 1-100,000 and 7-10,000’s. This yields 6+1=7-100,000’s and finally 1-1,000,000. The final answer is 1,771,561.

**Closure Notes**

Students will summarize their understanding of how to use Pascal’s Triangle to implement the Binomial Theorem. Students will recognize that the Binomial Theorem is a special identity that can expand any polynomial of the form *(a + b)n* without having to perform the repeated multiplication factor by factor. An important concept in this investigation is that students are able to generalize the theorem by making appropriate substitutions for *a* and *b* in the identity.

**Vocabulary**

Array

Binomial

Binomial Theorem

Coefficient

Combinations

Pascal’s Triangle

Power

Subsets

**Resources and Materials**

**This is an optional investigation for all students. If time permits STEM intending students should do it.**

Activity 3.4.1False Advertising: How Many Pizzas Can They Make?

Activity 3.4.2 Pascal’s Pizza Parlor!

Activity 3.4.3Pascal’s Triangle and the Binomial Theorem

Supporting Australian Mathematics Project – Binomial Theorem (<http://www.amsi.org.au/ESA_Senior_Years/PDF/Thebinomialtheorem1c.pdf>)

Teacher Notes for Activity 3

**Teacher Notes for Activity 3.4.3**

For the purpose of this course, the statement of the Binomial Theorem has been defined in terms of the entries in Pascal’s Triangle. The statement provided using the notation *cn,i* is not necessary to understand the basic application of the theorem and should be introduced only for the most sophisticated students.

The expansions of the binomials are shown below. Students should do the first 5 powers of (a+b) by hand and then, if possible, use a CAS such as TI-NSpire or GeoGebra CAS to calculate higher powers to confirm applications of Pascal’s Triangle.

(a+b)0 = 1

(a+b)1 = a + b

(a+b)2 = a2 + 2ab + b2

(a+b)3 = a3 + 3a2b + 3ab2 + b3

(a+b)4 = a4 + 4a3b + 6a2b2 + 4ab3 + b4

(a+b)5 = a5 + 5a4b + 10a3b2 + 10a2b3 + 5ab4 + b5

The CCSSM proposes possible proofs of the Binomial Theorem. One approach would be to use a combinatorics approach to explain why the entries in Pascal’s Triangle can be found by adding the adjacent entries in the previous line to obtain the entries in the next line. Another approach would be to use proof by induction. However, the proof by induction approach would necessitate the introduction of the concept and notation of combinations of n objects taken r at a time, a topic that is beyond the targeted population for this investigation. A link to a website supported by Harvey Mudd College for the formal proof of the Binomial Theorem is: <https://www.math.hmc.edu/calculus/tutorials/binomial_thm/induction.html>.

Combinatorics Explanation:

The number of subsets of r elements taken from a set of n+1 elements can be shown to be the sum of r elements taken from a set of n elements plus the number of subsets of r–1 elements taken from a set of n elements. The explanation is that if you add a new element to the set of n elements, then all the subsets with r elements from the original set of n elements would still be subsets of r elements in the new set with n+1 elements. Additionally, if you take all the subsets with r–1 elements and include the new element from the set with n+1 elements, you would now have a subset of r elements from the set of n+1 elements. Therefore, the sum of the subsets of r elements taken from n elements plus the subsets of r–1 elements taken from the set of n elements equals the number of subsets of r elements taken from the set with n+1 elements, exactly what adding the two entries from the previous line in Pascal’s Triangle represents.

Powers of 11 :

The example with the powers of 11 follow a simple pattern until you reach 115 when the digits no longer are the numbers in Pascal’s Triangle, since they now surpass a single digit. However, students should notice that the terms of the Binomial expansion form powers of 10 which represent the place values of the answer. Therefore, when the entry in Pascal’s triangle surpasses 10, the entry represents how many of that place value you have. Therefore, you put down the 1’s digit in the entry as the digit in that place value and carry the remaining portion of the number into the next place value. Doing this for each place value starting with the 1’s column, can find the expansion of the power of 11. The example below demonstrates the process.

115 = (10 + 1)5 = 1•105•10 + 5•104•11 + 10•103•12 + 10•102•13 + 5•101•14 + 1•100•15

115 = (10 + 1)5 = 1•105 + 5•104+ 10•103+ 10•102+ 5•101+ 1•100

Entering the results in the columns of a base 10 number we obtain the following result:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| 105’s | 104’s | 103’s | 102’s | 101’s | 100’s=1’s |
| 1-100000 | 5+1 =  6-10000’s | 10-1000’s +1=11-1000’s  put 1-1000 & carry 1-10000 | 10-100’s=1-1000 & 0-100’s  carry 1-1000 | 5 | 1 |
| 1 | 6 | 1 | 0 | 5 | 1 |

Or 115 = 161,051