**Activity 2.2.4 Absolute Value,** $\left|x\right|, \sqrt{x^{2} } and \pm x$

Start by putting your pencil at 0 on the number line. Then move your pencil and place a dot at

-3.



1. a. When you started at zero, in what direction did you move to place the dot at -3?

b. How many spaces did you move?

Notice that we can think of any number on the number line as having two attributes:

* Direction from zero, that is indicated by a + or – sign.
* Distance from zero, that is called its magnitude

The absolute value sign around a number or expression indicates the size (magnitude) of the number without regard to its sign. As you saw in Unit1 Investigation 4, the undirected distance between the two points 0 and -3 can be written as $\left|-3-0\right|$, which equals 3; or as $\left|0-(-3)\right|$, which also equals 3. The distance between two points is the same number whether you move left or right. The distance between two distinct points is a positive number. Symbolically, we can write the distance between two points *a* and *b* as $\left|a-b\right| or \left|b-a\right|$.

Example: What is the distance from point 3 to point 8?

Answer: $\left|8-3\right|$ that equals $\left|5\right|=5$

Example: What is the distance from point 8 to point 3?

Answer: $\left|3-8\right|$ that equals $\left|-5\right|=5$

On the other hand, the *directed distance* moving right from point 3 to point 8 is: 8 – 3 = 5. Whereas the directed distance moving left from 8 to 3 is: 3 – 8 = -5. The directed distance will be positive if you move right and negative if you move left.

1. Simplify the following expressions:

a. $\left|-3\right|$ b. $\left|3\right|$ c. $\left|5\right|$ d. $\left|-5\right|$

The absolute value bars act like a grouping symbol. When simplifying an absolute value expression, simplify the expression inside the absolute value bars first, and then take the absolute value of the result.

1. Simplify the following expressions:

a. $\left|3-5\right| $b. $\left|5-3\right| $c. $\left|-5+3\right| $d. $\left|5-5\right|$

After you take the absolute value of the inside number, multiply the result by the coefficient if there is one:

1. Simplify the following expressions:

a. $7\left|-5\right| $b. $-1\left|5\right| $c. $-\left|-3\right| $d. $-\left|7\right|$

1. Now that you know how to find the absolute value of a particular number, let’s generalize taking an absolute value of any number *x*. What do you think $\left|x\right|$ is equal to?
2. Be careful with this. You saw that $\left|+3\right|=3$ and$ \left|-3\right|=3$, but does $\left|x\right|=x$?
3. Try substituting 7 in for *x* in $\left|x\right|=x$. Do you obtain a true statement?
4. Try substituting -7 in for *x* in $\left|x\right|=x$. Do you obtain a true statement?
5. If *x* represents a negative number, how do you make it positive?
6. Explain in words how to take the absolute value of *x* if *x* is a positive number or zero.
7. Explain in words how to take the absolute value of *x* if *x* is a negative number.

Here is one way to summarize what you wrote in part d and e:

The absolute value of a number *x* $=\left\{\begin{array}{c}the same number x if x is 0 or a positive number\\the opposite of the number x if x is negative\end{array}\right.$

1. Write a piecewise definition of absolute value by filling in the blanks.

Use the symbols: *x*, -*x*, $\left|x\right|$, and 0.

**Graphs of Absolute Value Functions by Plotting Points and by Transformations**

1. Graph $y=\left|x\right|$ by plotting points. Pick at least five values for *x* – two positive, two negative and zero – and fill in the table. Identify the coordinates of the vertex.

|  |  |
| --- | --- |
| *x* | $$y=\left|x\right|$$ |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

1. Is the function $y=\left|x\right|$ an even function, an odd function or neither?
2. Notice that the graph consists of parts of two lines. An equation of the ray in quadrant II for negative *x* values is $y=-x$. What is an equation for the ray in quadrant I?

Absolute value functions consist of two pieces, so they are piecewise defined functions.

1. What is the smallest value that $y=\left|x\right|$ can be?
2. Consider $f(x)=\left|x-3\right|$. What should *x* be so that $\left|x-3\right|$ is the smallest number it can be?
3. Now pick some points less than 3 and greater than 3 to fill in the table and plot the graph of $f(x)=\left|x-3\right|$.

|  |  |
| --- | --- |
| *x* | $$y=\left|x-3\right|$$ |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

1. Use what you learned about translations in Unit 1 to describe how to graph $f\left(x\right)=\left|x-3\right|$ if you know the graph of $f(x)=\left|x\right|$.
2. Graph the following functions on the same coordinate plane. Write the coordinates of the vertex for each function.

$g(x)=\left|x-(-4)\right|$

$h\left(x\right)=\left|x-\left(-4\right)\right|-2$

1. Graph $j\left(x\right)=\left|x+2\right|-3$. Write the coordinates of the vertex.



1. Graph $k\left(x\right)=\left|x-1\right|+2$. Write the coordinates of the vertex.



1. Graph $m\left(x\right)=-\left|x\right|$. Write the coordinates of the vertex.



**Solving Absolute Value Equations**

You can use the definition of absolute value or graphing to solve equations involving absolute value.

**Tolerance for Manufactured Items**

Have you seen someone drop his or her smart phone and watch helplessly as the glass cracks? To solve the problem, an inventor created a new design for a protective phone cover. She hires a manufacturing company to make the covers so she can sell them. However, every manufacturing process has some error. This particular cell phone cover is designed to have a width of 600 mm. The inventor determines that the flexible protective cover will fit the cell phone if the width of the cover is at most 1.5 mm too large or too small. Use an absolute value equation to tell what is the smallest and what is the largest width that she can accept from the manufacturer?

*Solution:*Note that the definition of absolute value tells us that $\left|x\right| $is the distance that some number x is from zero. Note $\left|x\right| $can be written $\left|x-0\right|$, and it means the difference between some number x and zero. In this situation, we want to find the distance between the width of the actual cell phone cover ‘w’ and exact measure of 600 mm. We can write$\left|w-600\right|$ to show how much error there is in the width of the cover. An equation that tells the most or the least width that is acceptable is:

$$\left|w-600\right|=1.5$$

You have probably already determined that the maximum width of the cover is 600 + 1.5 and the minimum is 600 – 1.5. Here is a number line representation:



$\left|w-600\right|=1.5$ can be written as -(*w* – 600) = 1.5 OR *w* – 600 = 1.5

 Solving for *w* in each: *w* – 600 = -1.5 *w* – 600 = 1.5

 *w* = 600 – 1.5 *w* = 600 + 1.5

 *w* = 598.5 mm OR *w* = 601.5 mm

Notice that one absolute value equation can be written as two equations without an absolute value sign. $\left|x-a\right|=c$ is equivalent to –(*x* – *a*) = *c* or *x* – *a* = *c* which is equivalent to *x* – *a* = - *c* or *x* – *a* = *c.*

This also follows the piecewise definition of absolute value: If the inside of the absolute value symbol is negative, take the opposite of the number. If the inside is nonnegative, just keep the nonnegative number.

1. A woodcutter receives an order to cut one cord of wood so the wood fits in a wood stove. The order asks that the length ‘x’ of each piece of wood meet the following specifications in inches:

$$\left|x-16\right|=2$$

1. From the equation, tell the ideal size log that the customer wants.
2. What is the maximum acceptable error?
3. Write two equations from this one absolute value equation (the first one is done for you.)

-(*x* – 16) = 2 OR

1. Solve each equation in part c.
2. What is the length of the largest acceptable log?
3. What is the length of the smallest acceptable log?
4. Solve the same absolute value equation $\left|x-16\right|=2$ by graphing the following two equations and finding the x-coordinates of their points of intersection: $y=\left|x-16\right| and y=2$.



1. A farm-to-table gardener is selling his string beans to a restaurant that wants beans that are 10 cm long, because the beans need to be large enough to have fully developed their taste, but not so large that they lose their tenderness. The restauranteur asks the gardener to be sure that the length of all the beans *b* meets the following criteria:

$$\left|b-10\right|=2.25$$

1. How much variation from 10 cm does the restauranteur allow for string beans?
2. Write two equations from this one absolute value equation:
3. Solve each equation.
4. What is the smallest acceptable bean length?
5. What is the largest acceptable bean length?
6. Solve by graphing the following two equations and finding the x-coordinates of their points of intersection:

$$y=\left|x-10\right| and y=2.25$$



1. Solve the following equations using any method:

Algebraic method: Write $\left|inside abs. value\right|=c$ as two equations:

-(*inside abs. value) = c*  OR *inside abs. value = c*

Then solve the two equations.

Graphical solution: Graph the system of equations:

 $y=\left|inside abs. value\right|$

 $y=constant$

Then find the *x*-coordinate of the point(s) of intersection.

1. $\left|x-28\right|=3$
2. $\left|2x-30\right|=6$
3. $\left|\frac{x}{3}-5\right|=3$
4. The teacher asked the class to solve $\left|x-20\right|=-5$. Susan blurted out, “You can’t. There is no solution!” What did Susan notice that led her to this conclusion before she even wrote anything down on paper?

**Absolute Value, Quadratic and Square Root Functions**

1. The quadratic function $f\left(x\right)=x^{2}$ and the absolute value function $g(x)=\left|x\right|$ have some similarities.
2. Are the two functions even, odd or neither?
3. Evaluate the following: *f*(2) = *f*(-2) = *f*(6) = *f*(-6) =

 *g*(2)= *g*(-2) = *g*(6) = *g*(-6 )=

We say that the functions are two to one (2-1), because two different inputs can give the same output. The functions are not 2-1 everywhere. At the vertex, they are 1-1.

1. Now let’s investigate the square root of a square and see how it relates to absolute value. Simplify the following:

a. $\sqrt{3^{2}}$ b. $\sqrt{(-3)^{2}}$

c. $\sqrt{7^{2}}$ d. $\sqrt{(-7)^{2}}$

e. $\sqrt{789^{2}}$ f. $\sqrt{(-789)^{2}}$

1. If you have a positive number, explain in words how to take the square root of a positive number *x* that is squared.
2. If you have a negative number, explain in words how to take the square root of a negative number *x* that is squared.
3. Fill in the blanks to summarize what you wrote in part g and h:

$\sqrt{x^{2}}$$=\left\{\begin{array}{c} \\_\\_\\_\\_\\_\\_if x is 0 or a positive number\\\\_\\_\\_\\_\\_\\_\\_if x is a negative number\end{array}\right.$

Note that the radical symbol indicates the principal square root of whatever is inside the radical, provided the radical expression is defined. There are actually two solutions to the equation $x^{2}=c$ (for *c* a non-zero constant) but the radical symbol indicates the principal square root of a number. Principal square root means the positive square root of the number.

For example, there are two square roots of 49: +7 and -7, because 72 = 49 and (-7)2 = 49. By convention, we take 7 to be the *principal square root* of 49. The radical symbol indicates just the principal (positive) square root of 49. We write: $\sqrt{49}$ = 7.

1. Write a piecewise definition for finding the square root of a squared number by filling in the blanks. Use the symbols: *x*, -*x*, $\sqrt{x^{2}}$ and 0.
2. Now compare this with the piecewise definition of the absolute value of *x*. What do you observe?

Hopefully you are convinced that $\sqrt{x^{2}}=\left|x\right|$, that means whenever we take the square root of a number squared it equivalent to taking the absolute value of the number. We have seen that $\left|x\right|$ = *c* , *c* > 0 can be written as two equations: *x* = -*c* OR *x* = *c*. A shorthand for this compound statement is *x* = ± *c*. The following are equivalent:

$\left|x\right|=c$ $\sqrt{x^{2}}=c$ $x=-c or x=+c$ *x* = $\pm c$

Why do we write $\pm \sqrt{c}$ when solving an equation of the type$ x^{2}=c$?

Example:

Solve: $x^{2}=9$

 $\sqrt{x^{2}}=\sqrt{9}$ Take the square root of both sides of the equation

 $\left|x\right|=\sqrt{9}$ Rewrite $\sqrt{x^{2}} as \left|x\right|$

 $x=-\sqrt{9}$ or $x=+\sqrt{9}$ Write the absolute value equation as two equations.

 (a shorthand for this is $x=\pm \sqrt{9}$)

 $x=-3$ or $x=+3$ Simplify $\sqrt{9}$ to be 3 ($\sqrt{9}$ is a positive number)

 (a short hand for *x*= -3 or *x* = +3 is $x=\pm 3$)

The two solutions to the equation $x^{2}=9$ are -3 and 3. They can be written $\pm 3$.

You often see the short cut for this work written this way:

Solve: $x^{2}=9$

 $\sqrt{x^{2}}=\sqrt{9}$

 $x=\pm \sqrt{9}$

 $x=\pm 3$ This means *x* = 3 OR *x* = -3

The solutions are -3 and 3.

1. Solve the following by taking the square root of both sides of the equation: (once you understand how to do this, you can skip steps, and just write the answer.)

a. $ x^{2}=25$ b. $x^{2}=\frac{64}{81}$ c. $x^{2}=8$