**Activity 2.3.1 Closure and Sets of Numbers**

**Practice with Decimals**

1. Fill in the blank with a >, < or = symbol to make a true sentence.

a. 0.333\_\_\_\_ $0.\overbar{3}$

b. 0.6667 \_\_\_ $\frac{2}{3}$

c. $\frac{3}{7}$ \_\_\_\_\_\_ 0.42857142857142…

d. $0.\overbar{9}$ \_\_\_\_ 0.99999

e. $0.\overbar{9}$ \_\_\_\_\_ 1

f. 0.23232222…..\_\_\_\_\_ $0.\overbar{23}$

g. 3.1415\_\_\_\_\_ π

h. 3.142 \_\_\_\_\_ π

1. Express the following rational numbers as a decimal. Show your long division work. Note that every rational number can be written as either a terminating or a repeating decimal.

a. $\frac{35}{4}$ b. $\frac{1}{3}$ c. $\frac{15}{11}$

d. Write the terminating decimal 1.23 as a repeating decimal by affixing infinitely many zeros after the digit “3”:

This shows that every terminating decimal can be written as a non-termination decimal that repeats with 0’s at the end.

1. Write the decimal 1.23 as the ratio of two integers (i.e., write in the form: $\frac{}{}$ ).
2. Irrational numbers are both nonterminating and non-repeating. Put the following numbers in the correct column: (the columns will not necessarily be the same length)

0.5, 3.0330333033330 … (Increase the number of 3’s in each block), 1.234444…, 0.$\overbar{66}$, 7.1254364 …(assume the digits continue without repeating), $7.1\overbar{232}$

|  |  |
| --- | --- |
| **Rational numbers**(can be expressed as a repeating decimal) | **Irrational numbers**(can be expressed as non-repeating, non-terminating decimals) |
|  |  |
|  |  |
|  |  |
|  |  |

Warning: a decimal number that follows a pattern in not necessarily a rational number.

**Sets of Numbers**

Write a definition for each of the following sets of numbers in words or by example or both for each of the following sets of numbers. Write the letter that is commonly used for the set, if there is one.

* 1. Natural Numbers (also called the Counting Numbers)
	2. Whole Numbers
	3. Integers
	4. Rational Numbers
	5. Irrational Numbers
	6. Real Numbers
	7. For the six types of numbers you just defined, make a Venn diagram showing what sets of numbers are subsets of each other and what sets do not overlap.

**The Desire for Closure**

In mathematics, we like to be able to get an answer. We want closure. If we try to operate on a set of numbers and can’t get an answer in that set, we might want to create a new set of numbers so that there is an answer. We say a set is closed under an operation if and only if an operation performed on any numbers in the set has an answer in the same set.

1. Is the set of natural numbers closed under addition? i.e. if you add any two natural numbers, do you always get another natural number?

Give an example if yes. Give a counterexample if not.

(Note: finding one example that works, that doesn’t prove that all examples will work. Proving that a statement is true for all numbers requires more work.)

1. If you subtract the two natural numbers, 16 and 5, (16 – 5), do you get another natural number?

If you subtract 5 and 16 (5 – 16), is the result a natural number?

Is the set of natural numbers closed under the subtraction?

What kinds of numbers are needed so that there is an answer to 5 – 16?

1. Are the natural numbers closed under multiplication?

Give an example if yes, or a counter example, if not.

Are the Integers closed under multiplication?

Give an example if yes, or a counter example, if not.

1. Are the natural numbers closed under division?

Give an example if yes, or a counter example, if not.

(Note: finding just one counterexample is all you need to prove a statement is false.)

1. Are the integers closed under division?

Give an example if yes, or a counter example, if not.

1. What kind of numbers do we need so that the natural numbers and the integers will be closed under division?
2. Are the rational numbers closed under:

Operation: Example, if yes. Counter example, if not.

Addition?

Subtraction?

Multiplication?

Division?

1. a. Find the length of the diagonal of a square that has sides equal to 1 cm. (Show your work.)
2. Is the length of the diagonal of a square with sides equal to 1 a rational number or an irrational number?
3. Is the length of this diagonal a real number?
4. Where on the number line will you place $\sqrt{2}$? Find out by squeezing $\sqrt{2}$ between two numbers. Start with squeezing $\sqrt{2}$ between two numbers that are perfect squares:

1 < 2 < 4. Take the square root of all three numbers and simplify: $\sqrt{1}<\sqrt{2}<\sqrt{4}$ to the nearest whole number, we know that $\sqrt{2}$ is between 1 and 2. 1<$\sqrt{2}$ < 2.

Continue estimating and checking your estimates to find a decimal approximation to $\sqrt{2}$ to the nearest thousandth. Show your thought process as you find two approximations of $\sqrt{2}$ to the nearest thousandth: one approximation that is too low and one that is too high.

On the number line, graph two estimates $\sqrt{2}$ of to the nearest thousandths. One estimate will be a bit high and the other a bit low.

$$\sqrt{2}$$

Note:

* Irrational numbers “fill in” the gaps between the rational numbers.
* The terminating decimal 1.41 is just an approximation for $\sqrt{2}$ ; it is too low an estimate since 1.412=1.9881, that is not equal to 2, it is too low.
* Together, the rational and the irrational numbers make up the entire number line.
* The real numbers consist of the rational numbers and the irrational numbers.
* There are more irrational numbers than there are rational numbers.
1. a. Find the square roots of these positive rational numbers:

$\sqrt{\frac{36}{25}}=$ $\sqrt{1}=$ $\sqrt{0.64}$ =

1. Give an example that shows that the set of positive rational numbers is not closed under the square root operation.
2. What set of numbers do we need to allow the square rooting of positive rational numbers?

**More Questions about Sets of Numbers**

1. True or False? If the statement is true, give an example; if it is false, give a counter example.

1. The sum of an integer and a whole number is always a whole number.
2. The sum of two rational numbers is always rational.
3. The product of two natural numbers can never be 0.
4. The square root of a natural number is always a real number.
5. The sum of an integer number and its opposite is always 0.
6. The sum of a rational number and an irrational number is always irrational.
7. The difference of two irrational numbers is always an irrational number provided the two numbers are not equal to each other.
8. The sum of two irrational numbers is always irrational.
9. The product of an irrational number and a non-zero rational number is always irrational.
10. a. Explain why the sum of any two rational numbers is always rational.
11. Explain why the product of two rational numbers is always rational.
12. Explain why the sum of a rational number and an irrational number is always irrational.
13. Explain why the product of a non-zero rational number and an irrational number is always irrational.

Challenge Problems

**Proofs About Operations and Sets of Numbers**

Example: Prove that the product of a non-zero rational and an irrational is irrational.

We will do a proof by contradiction. We shall assume the product of a rational and an irrational is RATIONAL. We will see that this assumption leads to an obviously false statement. Then we conclude that the assumption must be false since it led us astray.

Assume that a rational number ‘a’ times an irrational number ‘b’ has a rational product ‘c’: ab=c.

Divide both sides of the equation by the rational number ‘a’ to get ab/a = c/a

That simplifies to b=c/a. Because c/a is a rational divided by a rational, and rationals are closed under division, then c/a is a rational number. However, b is an irrational number. That leads to the false statement that an irrational number ‘b’ equals a rational number ‘c/a’.

Since that is impossible, our original assumption that “ab” is rational must be false.

Therefore the product of a rational and an irrational is irrational.

Now you try it: Prove that the sum of a rational and an irrational number is irrational.