**Unit 6: Investigation 1 (3 Days)**

**Polygons and Polyhedra**

***CCSS:***

***6G4*** Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

***7G6*** Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

***8G9*** Know the formulas for the volumes of cones, cylinders and spheres and use them to solve real-world and mathematical problems.

***G-GMD.4***. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects

**Overview**

This investigation extends the vocabulary used to describe polygons to polyhedra. The term polyhedron is defined. Polyhedra are classified as convex vs. non-convex, prisms, pyramids, regular and semi-regular polyhedra. We use an informal proof to prove that there are only five regular polyhedra called the Platonic solids. Students explore polyhedra and make a conjecture about relationships between the numbers of faces, edges and vertices (Euler’s formula).

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Students will be able to construct models of prisms, pyramids, regular and semi-regular polyhedra from concrete materials.
* They will use strategies to count the numbers of vertices, edges and faces.
* They will determine a relationship between these numbers (Euler’s Formula).

**Assessment Strategies: How Will They Show What They Know?**

* **Exit Slip 6.1.1** asks students to classify shapes using Venn Diagrams.
* **Exit Slip 6.1.2** involves the application of Euler’s formula.
* In the **Journal Entry** students will describe their strategy for counting edges, faces, and vertices.

**Launch Notes**

During the closing ceremony of the 2000 Sydney Olympics the Aboriginal singer Christine Abu sang a song “My Island Home” which was performed on a flexible dodecahedral stage. You can see it here : [http://m.youtube.com/watch?v=kDyc-T2AYt4](https://webmail.ccsu.edu/owa/redir.aspx?C=eKGlQE3MTUaBwxEd8c0VQJZHhMIl0tFIfo8mIdBnsDXMPvcZ3FugYQUGrbGovpanFb7WwWq0HhE.&URL=http%3A%2F%2Fm.youtube.com%2Fwatch%3Fv%3DkDyc-T2AYt4). The real action takes place between 2:30 and 5:00 minutes. As a finale the image of the earth is projected on the revolving stage. The staging was designed by Australia’s best-known stage designer Brian Thomson. He studied architecture before he became a set designer.

The magazine “Milesago” has described it as follows:

“The dodecahedron shape was both symbolic, and a striking sculptural form, but it was also a brilliant piece of multi-functional stage design which was used in many ways throughout the show. Structurally, the top half of the dodecahedron was made as a single piece, effectively making it a lid which could be raised and lowered into place on top of the bottom half on a cable. The bottom half was a masterpiece of ingenuity. Each of the five “leaves” of the bottom half could be individually raised or lowered hydraulically, opening up like a flower or closing to become a cup, and each leaf could be used as a ramp or an elevated platform. The floor panel had a concealed hydraulic lift in the center, which dropped through the stage floor; this enabled cast members to enter the dodecahedron without being seen when the leaves were raised. Finally, the entire apparatus was mounted on a turntable, which could be rotated through 360 degrees in either direction. The inside of the dodecahedron was covered in a neutrally coloured fabric to provide non-skid surface for the performers; the outside was painted silver to reflect the lighting and projections.

The dodecahedron shape was no doubt inspired in part by Brian’s architectural studies, but the object was also eminently appropriate for the show, being rich in symbolism. The dodecahedron is a “Platonic solid” composed of 12 pentagonal faces. The Platonic solids, also called the regular solids or regular polyhedra, are convex polyhedra with equivalent faces composed of congruent, convex regular polygons with equal sides and equal angles. There are exactly five such solids: the cube, dodecahedron, icosahedron, octahedron, and tetrahedron, as was proved by Euclid in the last proposition of the *Elements*. The Platonic solids are sometimes also called "cosmic figures".

The Platonic solids were known to the ancient Greeks, and were described by Plato in his *Timaeus* ca. 350 BC. Plato equated these polyhedra with the Greek elements: the tetrahedron with fire, the cube with earth, the icosahedron with water, the octahedron with air. Plato assigned the dodecahedron the role of *"the fifth construction, which the god used for embroidering the constellations on the whole heaven"*. In 1596 Johannes Kepler published a tract called *The Cosmic Mystery* in which he envisioned the universe as consisting of nested Platonic Solids whose inscribed spheres determine the orbits of the planets, all enclosed in a sphere representing the outer heaven.

The dodecahedron is representative of the universe, and the number of faces also alludes to the twelve disciples of Jesus, the twelve months of the year and the twelve houses of the zodiac. In addition the twelve faces of the dodecahedron are pentagons. The pentagon is the polygon that contains the **golden ratio**, strengthening the association this figure has with the cosmos, and joining the vertices of a pentagon can also trace out the pentagram or pentangle.”

The figures below (taken from [http://www.britishmuseum.org](http://www.britishmuseum.org/)) shows a faceted dice with letters of the Greek alphabet from alpha to upsilon likely from the first century BCE.



**Teaching Strategies**

**Group Activity**

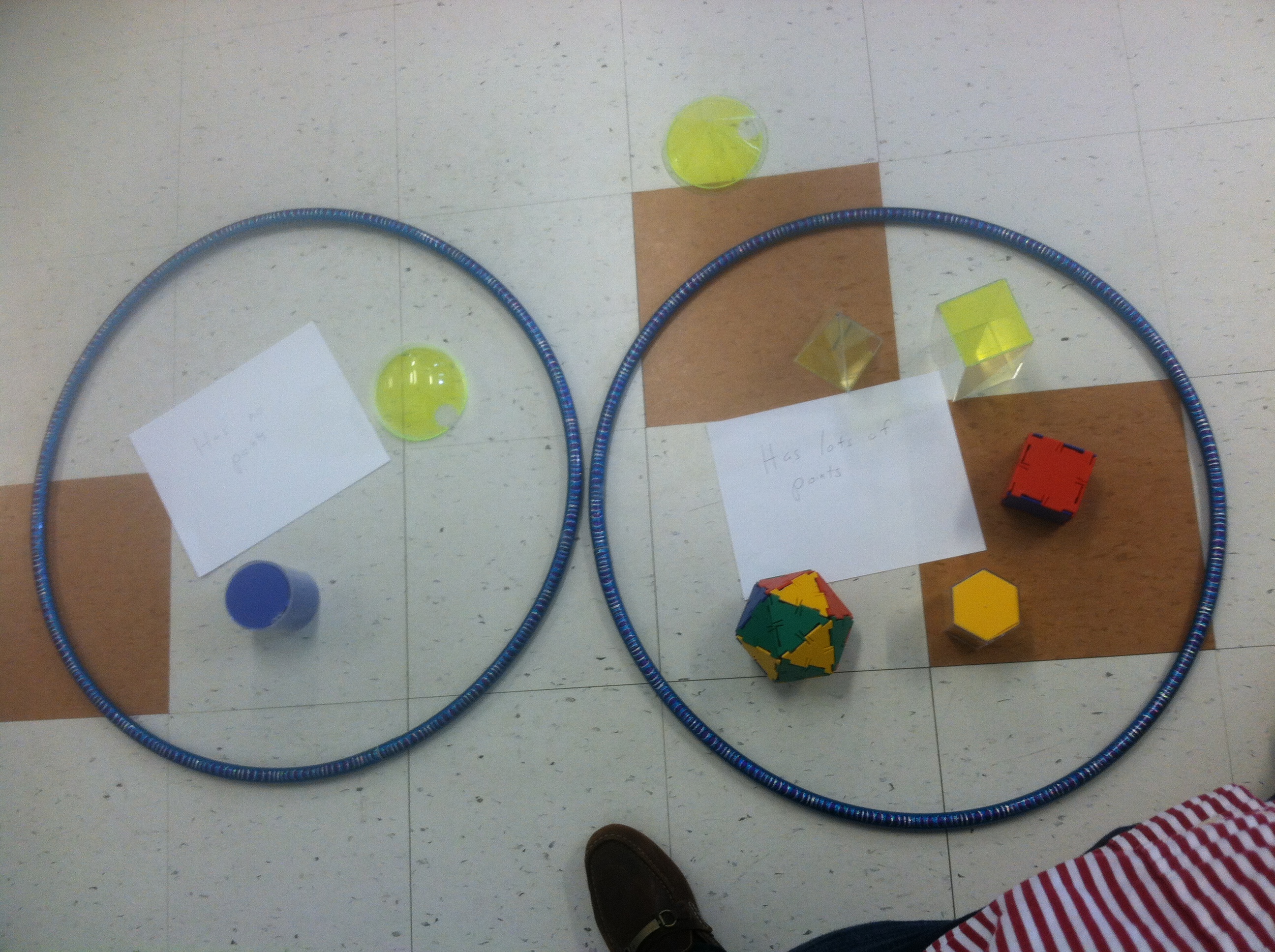
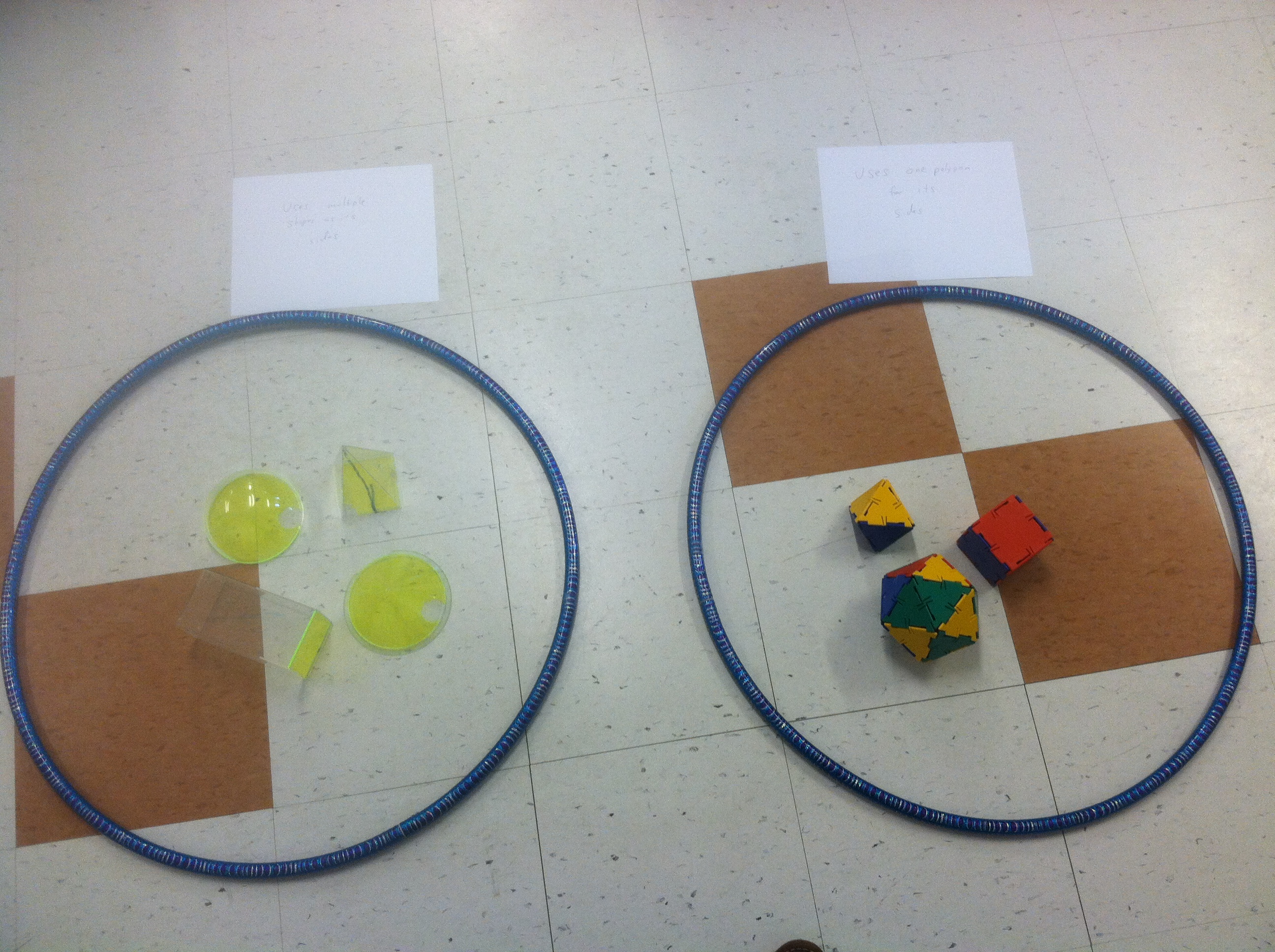
To initiate a productive conversation about the properties of solids, have students work in groups of three or four on **Activity 6.1.1**.

Start with **Activity 6.1.1 Introduction to Classifying Solids.** In this activity students classify a number of solids by the properties they possess. Have students work in groups. Give each group a collection of three-dimensional shapes. Physical models are best but printed pictures will work. Have them list five properties of each shape on the activity sheet.

Suggested shapes: cube, regular tetrahedron (triangular pyramid), triangular prism, rectangular prism, regular dodecahedron, regular octahedron, cylinder, cone, sphere. Not every group needs to have exactly the same set of shapes.

The next step is to place these shapes in a Venn diagram. You may give each group two hula-hoops or have them draw circles on butcher paper. It is important to allow the freedom of have the circles overlap if they need or be disjoint if needed. Ask each group to identify two distinctive characteristics (ideally from the descriptions the created on the handout) so that every one of there shape has at least one of those characteristics. Then place each shape in the appropriate region of the Venn diagram.

Here are some examples. In the figure on the left, students made two categories: “All sides are the same polygons” and “the sides are made of different shapes.” On the right, students made two categories: “Has no corners” and “Has lots of corner.” Notice the cone with a single “corner” (a.k.a. vertex) was placed outside each circle.



Have each group present their findings to the class. As an extension you may want to have some groups try the same exercise with three circles.

Once students have presented their findings you may introduce the formal vocabulary we will be using throughout this unit. If the surface is made up entirely of polygons, the figure is a **polyhedron**. (plural: polyhedra). The polygons are called **faces**, their sides are the **edges** of the polyhedron, and three or more faces meet at a point called a **vertex**.

You may want to point out to students that the prefix “poly-“ (from classical Greek) means “many.” A polygon is a two dimensional figure with many sides, whereas a polyhedron is a three-dimensional figure with many faces.

**Prisms** and **pyramids** are special cases of polyhedra. A polyhedron is said to be **convex** if its surface does not intersect itself and the line segment joining any two points of the polyhedron is contained in the interior or surface. **Regular** and **semi-regular** polyhedra will be introduced in **Activity 6.1.3**.

Some of the figures examined in this activity were not polyhedra. Students may already be familiar with **cylinders**, **cones**, and **spheres** from previous courses.

Following this activity you may give students **Exit Slip 6.1.1**.

In **Activity 6.1.2 Isometric Perspective Drawing,** students use an isometric grid to create perspective drawings of polyhedra**.** There are two versions of this activity. **Activity 6.1.2a** requires the use of isometric grid paper (provided with the activity sheet) and colored pencils. **Activity 6.1.2b** uses GeoGebra.

Both activities begin by having students draw a single cube and then two cubes of different colors stacked on top of each other. Then challenge students to draw a non-convex polyhedron formed by smaller cubes snapped together. You may differentiate by giving different students or pairs of students configurations of various difficulties. One example consisting of 11 cubes with 6 different colors is shown on the activity sheet.

A second set of exercises involves drawing prisms or pyramids of various dimensions. You may assign different polyhedra to different individuals or pairs, again based on how challenging you think they will be.

Allow time for students to share their work with the entire class.

In **Activity 6.1.3 Regular and Semi-Regular Polyhedra** students prove that there are exactly five Platonic solids (regular polyhedral) and are introduced to semi-regular polyhedra. (In Unit 8 there is an extension of this activity in which students classify all 13 semi-regular polyhedra and prove no others exist). A good way to begin this exploration is by using Magformers (magnetic building construction sets).

In **Activity 6.1.4 Euler’s Formula** students investigate the relationship between the number of faces, edges and vertices for a polyhedron. There are two versions of this activity. **Activity 6.1.4a** should be accessible for most students. In **Activity 6.1.4b** students are given an extra hint to aid in the discovery of the formula in two questions instead of being asked to make a sketch, the sketch is provided for them.

**Differentiated Instruction (For Learners Needing More Help)**

Some students will need to count vertices, edges and faces directly and will have difficulty keeping track of what has been counted. Use colored dot stickers to assist in the counting. Another approach is to use the NCTM Illuminations interactive Geometric Solids site: <http://illuminations.nctm.org/Activity.aspx?id=3521>

This interactive model lets students mark and count vertices, edges and faces and helps students keep track of the count.

Following **Activity 6.1.4** you may give **Exit Slip 6.1.2.**

**Activity 6.1.5 Schlegel diagrams** providesan optional extension.Students work in small groups with “shadows” of wireframe polyhedra to get an intuitive understanding about Schlegel diagrams, after which they will attempt to pair the Schlegel diagram and its corresponding polyhedron.

“Polytope” is the general term to describe a geometric object that encloses space. So far we have talked about polygons that enclose regions of two-dimensional space in the plane and polyhedra that enclose regions of three-dimensional space in their interior. Technically a Schlegel diagram is the projection of a polytope from a *d*-dimensional space into *d* – 1 dimensions. Schlegel diagrams are named for Viktor Schlegel who invented this type of diagram in 1883.

In this activity *d*  = 3. We will be projecting 3-dimensional polyhedra onto a 2-dimensional plane. Question 6 there is an extension of this work in which we take a 4-dimensional hypercube or **tesseract** and project it onto 3-dimensional space. This allows us to explore the numbers of edges, vertices, faces and cubes in the four-dimensional figure. Four dimensional figures will be studied in greater detail in Unit 8.

**Differentiated Instruction (Enrichment)**

Schlegal diagrams are used in some proofs of Euler’s formula. One that may be accessible to your more advanced students is found here: <https://plus.maths.org/content/eulers-polyhedron-formula>.

**Closure Notes**

Have students compare and contrast the concepts of polygon and polyhedra. In the process you will review the vocabulary: vertex, side, edge, face, surface, and regular.

**Journal Prompt**

Describe the strategy you used to count the number of faces, vertices, and edges of a polyhedron. Look for students to demonstrate that they understand the distinction between these three parts of a polyhedron.

**Vocabulary**

concave

cone

convex

cube

cylinder

dodecahedron

edge (of polyhedron)

face (of polyhedron)

hexahedron

icosahedron

octahedron

polyhedron

prism

pyramid

regular polyhedron

Schläfli symbol

Schlegel diagram

semi-regular polyhedron

sphere

tesseract

tetrahedron

vertex (of polyhedron)

**Theorems**

**Regular Polyhedron Theorem:** There are exactly five regular polyhedra. (Informal proof)

**Euler’s Formula**: For any convex polyhedron, *V* + *F* = *E* + 2, where *V* is the number of vertices, *F* is the number of faces, and *E* is the number of edges. (not proved)

**Resources and Materials**

Stella is a software program that explores nets of polyhedra and can assist in the making of many sophisticated models (<http://www.software3d.com/>). Magnus Wenninger has used this software to assist in recent models. This site has mobile apps for Android based devices.

Pedagoguery Software (<http://www.peda.com/poly/> ) provides shareware software to explore nets, Schlegel diagrams and three dimensional animations of semi-regular and regular polyhedra and other polyhedra.

Jeffrey Weeks KaleidoTile encourages informal experiments with semi-regular and regular polyhedra and their symmetries. ( <http://geometrygames.org/KaleidoTile/index.html>).

The NCTM publication *Polyhedron Models for the Classroom*, by Magnus J. Wenninger (1966) is a classic reference for this unit.

H.S.M. Coxeter’s *Introduction to Geometry* (1961), Wiley Press, Chapter 10 is another classic resource that provides the teacher or advanced student with a discussion of the Platonic solids which includes Schläfli symbols, Schlegel diagrams and the dihedral angles between the faces. It also includes a proof of Euler’s Formula and a formal proof that there are just five convex regular polyhedra.

*The Symmetries of Things*, by John Conway, Heidi Burgiel and Chaim Goodman-Strauss (2008) A. K. Peters, includes a much more sophisticated view of polyhedra through the properties of symmetry for teachers and very advanced students.

An interesting website for exploring polyhedra is found at : <http://world.mathigon.org/Polygons_and_Polyhedra>

This website is beautiful, but alas under construction, within this vast resource is a section on origami constructions <http://www.mathigon.org/origami/> As part of the explorations, you should download the folding instructions from here.

**NCTM Illuminations:**

These would be particularly helpful for students who are less able:

Popcorn Anyone <http://illuminations.nctm.org/Lesson.aspx?id=2927>

Fishing for the best Prism <http://illuminations.nctm.org/Lesson.aspx?id=2911>

The interactive Geometric Solids (<http://illuminations.nctm.org/Activity.aspx?id=3521>)

explores counting edges, faces, and vertices with marking tools on virtual models.

The Isometric Drawing Tool ( <http://illuminations.nctm.org/Activity.aspx?id=4182>) uses a JAVA applet to let students make virtual models of rectangular prisms.

**National Library of Virtual Manipulatives:**

Three modules may be of interest **Platonic Solids,** **Platonic Solids – Duals**, and **Platonic Solids Slicing**. These modules require the use of JAVA applets.