**Unit 7: Investigation 2 ( 4 Days)**

**Theoretical and Experimental Probability**

**Common Core State Standards**

* **CP-1.** Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events (“or,” “and,” “not’).
* **S-CP 9.** (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

**Overview**

Investigation 2 introduces two approaches to assigning probabilities: the **classical approach** and the **frequentist approach**. The classical approach requires that individual outcomes in a finite sample space are equally likely. The probability that an event *A* occurs is computed from the following ratio: (number of outcomes in *A*)/(number of outcomes in the sample space). Assigning probabilities using the frequentist approach requires computation of relative frequencies (or proportions) based on many, many trials of a random process.

Students use both the classical approach and frequentist approach to assign probabilities to events associated with flipping a coin and rolling a die. Simulation is introduced to estimate probabilities using the frequentist approach. Probability estimates from simulations are compared to probabilities calculated from an area probability model. Counting methods—the Fundamental Counting Principle, permutations, and combinations—are developed to aid in calculating probabilities when outcomes in the sample space are equally likely.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Create a probability model in which all simple outcomes in a finite sample space are equally likely.
* Use counting techniques (Fundamental Counting Principle, combinations, permutations) to determine the number of ways an event can occur. Then determine the probability that the event occurs.
* Estimate the probability that an event will occur from the relative frequency that the event occurs in many trials.
* Represent a sample space and events with an area probability model and use it to find probabilities.
* Use simulation to estimate the probability that an event will occur.

**Assessment Strategies: How Will They Show What They Know?**

* **Journal Entries 1 and 2** revisit two of the situations discussed in the launch of Investigation 1. Students are asked to interpret the meaning of the probabilities in these situations and to discuss how these probabilities could affect decisions.
* **Exit Slip 7.2.1** presents students with data from a survey question. Students are asked to assign probabilities to the possible responses to the question and then use the rules of probability covered in Investigation 1 to calculate probabilities of other events.
* **Exit Slip 7.2.2** asks students to use the counting techniques covered in this investigation to find the number of outcomes in a sample space or certain events. Then students use their results to calculate several probabilities.

**Launch Notes**

In Investigation 1 probabilities of events were given to students so that they could calculate the probabilities of other events. However, there was no attempt to explain how the assigned probabilities were determined. For example, how does a meteorologist determine the probabilities that certain weather events will occur and what do those probabilities mean? How did the Connecticut State Lottery determine the probability of winning the Mega-millions jackpot? These questions will be addressed in Investigation 2.

To begin this activity, flip a coin in front of the class but don’t show the students the outcome. Have them describe the sample space. Then ask them what probability they would assign to your just having flipped a tail. Encourage them to report this probability as a number between 0 and 1. Take a few minutes to review the concept of a random process (or random phenomenon, which is the term used in the video that is part of this launch). In this case, students know the possible outcomes of the random process—you either flipped a head or a tail— but students can’t predict for sure the outcome of this particular flip. Show students the result of your coin flip. Then, flip the coin another four times and record the outcomes from the five flips. Ask students to look for a pattern that would allow them to predict for certain what the result of the sixth flip would be. The point here is that in a random process the individual outcomes in a few trials are impossible to predict for certain.

Next, have students watch the following video: “Introduction to Probability” from Unit 18, *Against All Odds: Inside Statistics.* The web address is [www.learner.org/courses/againstallodds](http://www.learner.org/courses/againstallodds). (If you don’t have a computer with Internet access in your classroom, assign the video for homework prior to this class.)

In this video a meteorologist discusses the process of assigning probabilities to particular weather events (such as snow or rain) and provides some background on the data collection that goes into forecasts. This video also discusses the process for determining the probability of tossing a coin and getting a tail and the probability of an asteroid hitting Earth.

**Teaching Strategies**

After students have viewed the video on probability, they should begin **Activity 7.2.1**.

**Activity 7.2.1 Assigning Probabilities** introduces students to two approaches to assigning probabilities: the classical approach**,** in which all outcomes in the sample space are equally likely, and the frequentist approach, which assigns probabilities to events based on relative frequencies over many, many trials.

**Materials:** At least one pair of dice per group. If dice are not available, then simulate rolling a die on a TI-84 as follows:

* Press MATH , highlight PRB, and press 5 (for rand(INT)
* Complete the command by pressing 1 , 6 ) .
* Press ENTER repeatedly to simulate rolling a die repeatedly.

The activity begins with a definition of a probability model.

**Probability model:** A mathematical description of a random process consisting of two parts: a sample space *S* and a way of assigning probabilities to events.

Students will use the classical approach and frequentist approach to create probability models associated with rolling a single die and rolling a pair of dice. The activity is split into two sections:

* Part I: Equally Likely Outcomes: For questions 1–4 probabilities are assigned under the equally-likely assumption (classical approach).
* Part II: Using Relative Frequencies to Approximate Probabilities: For questions 5–8 probabilities are assigned by observing many trials and recording the relative frequencies of certain outcomes or events (frequentist approach).

**Group Activity**

At least part of **Activity 7.2.1** *Assigning Probabilities*should be done as a group activity.

Question 1 asks students to use the rules of probability from Investigation 1 to justify that the probability of getting a head when flipping a fair coin is ½. You may want to give students a few minutes to come up with an answer to question 1 individually and then discuss the answer as a whole class. Questions 2–4 can be done independently or in groups.

Questions 5 and 6 should be done in small groups of 2 or 3 students. In order to answer these questions, groups need to roll a die or a pair of dice 100 times. If students are in pairs, one student can roll the dice and the other can record the outcomes. If students are in triples, two students can roll the dice and the third can record the outcomes.

For question 7(a) groups need to combine their results into a set of class data and record the class data in Tables 7 and 8 on the activity sheet (copies of these tables appear in the PowerPoint presentation: *Tables*, in which the class data can be recorded). Students then use the class data to answer the remainder of question 7.

If there is insufficient time to complete the activity in class, question 8 can be assigned as homework.

Journal Entries 1 and 2 may be assigned at any time following Activity 7.2.1

**Journal Entry 1**

Discuss the meaning of a 70% chance of a foot or more of snow compared to a 10% chance of a foot or more of snow. How might this difference in the weather forecast affect your decision on whether or not to study for a test scheduled for the next day? Look for students to note that a 70% chance of snow means that snow is more likely (in fact, 7 times more likely) than a 10% chance of snow. Students might be willing to risk not studying for a test if the chance of a foot or more of snow is 70%, but not if there is only a 10% chance.

**Journal Entry 2**

Discuss the meaning of the following: The probability of winning the Mega-millions jackpot in the Connecticut State Lottery is 1/258,890,850. How might this probability affect your decision of whether or not to buy lottery tickets?

Look for students to note that it is very unlikely for them to win the Mega-millions jackpot. Perhaps they would be better off saving their money.

**Activity 7.2.2 Approximating Probability Using Data** gives students an opportunity to work with an area probability model and the frequentist approach for assigning probabilities. Each group of students is given a box and is asked to create four different geometric shapes. There are two required shapes, a circle and triangle and students are free to choose the other two shapes. After creating their shapes, students must glue them to the bottom of the box. The box bottom represents the sample space and the shapes events. Probabilities of events are determined by the proportion of the sample space’s area that is covered by the shapes. Students compare their theoretical probabilities (calculated from ratios of areas) with probabilities estimated from simulations in which students drop beans into the box and record the relative frequency of beans that land on various shapes (or combinations of shapes). Students discover that the simulation results improve as the number of beans being dropped increases.

**Materials for Activity 7.2.2:** One rectangular box will be needed for each group. In addition, groups will need colored paper, scissors to cut out shapes, a compass for drawing circles, a ruler, tape or a glue stick, a container of small beans (or beads), and a paper cup (for holding a specific number of beans).

To save class time on **Activity 7.2.2**, each group should prepare their box and calculate the areas of each of their shapes outside of class. Then, only the containers of beans per group would need to be prepared for the in-class activity.

**Group Activity**

Students should work on Activity 7.2.2 *Approximating Probability Using Data* in small groups of 2 to 4 students.

**Differentiated Instruction (For Learners Needing More Help)**

Adapt **Activity 7.2.2** either by reducing the number of shapes or by choosing shapes for which the calculations of their areas are easy to determine.

**Differentiated Instruction (Enrichment)**

Adapt **Activity 7.2.2** either by increasing the number of shapes or by choosing shapes for which finding the areas is more challenging.

**Activity 7.2.3** **Estimating Probabilities from Survey Data** gives students data on the responses to a survey question from two different surveys. Students calculate relative frequencies of the possible responses. Given that the samples were drawn in such a way that they are representative of the populations, the relative frequencies of the responses can be used as estimates of the probabilities (frequentist approach). After computing the probability estimates, students use the General Addition Rule and Complement Rule to find probabilities of several events.

**Activity 7.2.3** can be assigned as homework. This activity provides good practice before students complete **Exit Slip 7.2.1**.

**Exit Slip 7.2.1** can be assigned after students have completed **Activity 7.2.3**.

**Activity 7.2.4 The Fundamental Counting Principle** reviews how to assign probabilities in cases where individual outcomes are equally likely. In order to count the number of outcomes in an event, students discover the Fundamental Counting Principle. Then they use the Fundamental Counting Principle to calculate the number of possible arrangements (permutations) of cards and colored jellybeans. In the last question, order is removed from the jellybean scenario and students determine the number of possible combinations of jellybeans.

**Fundamental Principle of Counting**

 If there are *n*1 ways of choosing one thing, *n*2 ways of choosing a second after the first is chosen, . . . , and *nk* ways of choosing the last item after the earlier choices, then the total number of choices for the sequence is $n\_{1}×n\_{2}×\cdots n\_{k}$

In **Activity 7.2.4** the concepts of permutations and combinations are introduced without using the terminology. Discuss the terms permutation and combination in the context of the jellybean scenarios in questions 5 and 6.

In question 5, six different colored jellybeans are placed in a jar. A certain number of jellybeans, say *k*, are selected and arranged in a row. In this scenario, we are dealing with permutations, ordered arrangements of the jellybeans. For example, the following two arrangements of all six jellybeans are different because the colors appear in a different order.

red, yellow, green, black, pink, blue

blue, yellow, red, green, pink, black

Challenge students to generalize their answers to question 5.

* Starting with 6 different colored jellybeans, how many ways can you randomly select *k* jellybeans and arrange them in a row? In other words, challenge students to come up with a formula that could be used to answer all parts of question 5.

Answer: (6)(5) . . . (6 – k +1)

* Next, generalize further. Starting with *n* different colored jellybeans, how many ways can you randomly select *k* jellybeans and arrange them in a row?

Answer: 

The following box summarizes the result of the discussion on question 5 above.

A **permutation** is an ordered arrangement of *k* items that are chosen without replacement from *n* items. The number of arrangements, denoted as , can be calculated using the following formula: *nPk =*$n\left(n-1\right)\left(n-2\right)\cdots \left(n-k+1\right)=\frac{n!}{\left(n-k\right)!}$

where $n!=n(n-1)(n-2)\cdots 2∙1$ and 0! = 1.

 In question 6, two jellybeans were selected from the jar and placed into a cup. In this case, all that mattered was the final collection of colors of the jellybeans and not the order. In this scenario, we are dealing with combinations, unordered sets of jellybeans. Review the answer to question 6(a) making sure to connect it to the answer to question 5(e). Then challenge students to answer the following question:

How many different cups of jellybeans are possible if three jellybeans are removed from the jar and placed into the cup? Here’s an outline of the thought process used to answer this question.

* From question 5(d), we know that if three jellybeans are randomly chosen and arranged in a row, there would be 120 different arrangements.
* For the number of different cups of three jellybeans, putting the arrangement of a red, blue, and green jellybean into a cup is the same as putting the arrangement of a blue, red, and green jellybean into a cup. All that matters is the combination of colors, not the order.
* For any three colors of jellybeans, there are 3! = (3)(2)(1) = 6 possible arrangements of those colors. So the number of combinations of 3 jellybeans chosen from 6 different colored jellybeans is 120/6 = 20.

Encourage students to list all 20 of the combinations. We’ve listed them below.

(1) red, yellow, green (2) red, yellow, black (3) red, yellow, pink (4) red, yellow, blue

(5) red, green black (6) red, green pink (7) red, green, blue (8) red, black, pink (9) red, black, blue (10) red, pink, blue (11) yellow, green, black (12) yellow, green, pink (13) yellow, green, blue (14) yellow, black, pink (15) yellow, black, blue (16) yellow, pink, blue (17) green, black, pink (18) green, black, blue (19) green, pink, blue (20) black, pink, blue

At this point students should be ready for a general discussion of combinations based on the box below.

A **combination** is an unordered selection of *k* items that are chosen without replacement from *n* items. The number of combinations, denoted as , can be calculated from the following formula: 

**Activity 7.2.5 Permutations and Combinations** provides amixture of problems in which students must determine which counting technique is appropriate – the Fundamental Counting Principle, permutations, or combinations.

You may want to give students some practice using the formulas for computing the number of permutations or combinations before starting **Activity 7.2.5**. Here are some examples.

1. 

2. 

3. 

4. 

You may also want to allow students to use TI-84 graphing calculators to calculate the number of permutations and combinations, especially when large numbers are involved. Here are instructions:

Permutation: To calculate :

* Enter the value for *n*.
* Press   MATH  PRB and select   2 (for nPr).
* Enter the value for *k* and then press ENTER .

Combination: To calculate :

* Enter the value for *n*.
* Press   MATH PRB and select   3 (for nCr).
* Enter the value for *k* and then press ENTER .

**Differentiated Instruction (For Learners Needing More Help)**

To help students differentiate permutations from combinations, place 5 books (which we will label A, B, C, D, and E) in a stack.

Task #1: Ask students to select two books and physically arrange them on a shelf (or on the top of a desk). Then they should list all possible arrangements and count them. Here’s the list:

 AB, AC, AD, AE, BA, BC, BD, BE, CA, CB,

 CD, CE, DA, DB, DC, DE, EA, EB, EC, ED

Ask students whether this is an example of combinations or permutations.

Students should verify that they have listed all possible arrangements using the formula for counting the number of permutations:  = (5)(4) = 20.

Task #2: Ask students to select two books and place them in a backpack. How many different ways can they select the books to place into the backpack? They should notice that since order doesn’t matter in this situation, half of the arrangements from Task #1 would be redundant. For example, AB and BA put the same books into the backpack. Here’s a list of the different sets of books that could be put into the backpack:

 AB, AC, AD, AE, BC, BD, BE, CD, CE, DE

Ask students whether this is an example of combinations or permutations.

Students should verify that they have listed all possible combinations using the formula for counting the number of combinations:  = 20/2 = 10.

**Differentiated Instruction (Enrichment)**

Extension to question 3, **Activity 7.2.5** *Permutations and Combinations*,on the Mega Millions game

If you match 4 out of 5 of the “white” numbers but not the Mega Ball “yellow” number, you win $500. What is the probability of winning $500?

Answer:

The number of ways 4 of the 5 first numbers on your lottery ticket can match the 5 selected “white” numbers is 5*C*4; the number of ways your 5th number matches the losing “white” numbers is 70. The number of ways your “yellow” number matches with the losing “yellow” numbers is 14. So, the number of ways of matching 4 out of 5 “white” numbers but not the Mega Ball “yellow” number is 5 × 70 × 14 = 4,900.

 *P*(winning $500) = $\frac{4900}{258,890,850}$ ≈ $\frac{1}{52,835}$

**Exit Slip 7.2.2** asks students todetermine the number of ways several events can occur and then to use this information to assign probabilities to these events. This will require calculations of permutations and combinations

**Closure Notes**

Reiterate that probability does not predict the outcome of a random process with certainty in the short term. However, over many, many trials a pattern emerges and probability does predict the long-run pattern with certainty. Even though you can’t predict the outcome of a random process for certain, you can use probability to help make decisions. Particularly if you plan to repeat a process, making decisions based on higher probabilities is a sensible strategy.

Review the three counting methods introduced in this investigation. Make sure that students recognize the formulas for computing the number of combinations and permutations in both their expanded notation and more compact notation. In addition, check that students understand factorial notation.

Tie the need for counting methods back to assigning probabilities. Note that this approach is only applicable when individual outcomes are equally likely. Fortunately, there are many situations in which individual outcomes are equally likely. This is true in games of chance that involve rolling a die or drawing cards from a deck of cards. On the other hand, the frequentist approach for assigning probabilities requires the calculation of relative frequencies based on many, many trials. The frequentist approach is useful in estimating probabilities based on data.

**Vocabulary**

**Combination:** An unordered selection of k distinct items that are chosen without replacement from a collection of n items. The notation for the number of combinations is *C*(*n*, *k*), , or , which is often pronounced “*n* choose *k*.” The formula for  is: 

**Factorial:** A notation used in computing the number of permutations or combinations. By definition$n!=n(n-1)(n-2)\cdots 2∙1$ for positive integers *n* and 0! = 1.

**Fundamental Principle of Counting:** If there are *n*1 ways of choosing one thing, *n*2 ways of choosing a second after the first is chosen, . . . , and *nk* ways of choosing the last item after the earlier choices, then the total number of choices for the sequence is $n\_{1}×n\_{2}×\cdots n\_{k}$

**Permutation:** An ordered arrangement of k distinct items that are chosen without replacement from a collection of n items. The notation for the number of permutations is P(n, k), or . The formula for  is: *nPk =*$n\left(n-1\right)\left(n-2\right)\cdots \left(n-k+1\right)=\frac{n!}{\left(n-k\right)!}.$

**Probability model:** A mathematical description of a random process consisting of two parts: a sample space *S* and a way of assigning probabilities to events.

**Relative frequency of an event:** The number of times an event occurs divided by the number of trials.

**Resources and Materials**

Activity 7.2.1 Assigning Probabilities

Activity 7.2.2 Approximating Probability Using Data

Activity 7.2.3 Estimating Probabilities from Survey Data

Activity 7.2.4 Fundamental Counting Principle

Activity 7.2.5 Permutations and Combinations

Journal Entry

Exit Slip 7.2.1

Exit Slip 7.2.2

Video: “Introduction to Probability” from Unit 18, *Against All Odds: Inside Statistics.* The web address is [www.learner.org/courses/againstallodds](http://www.learner.org/courses/againstallodds).

Sufficient pairs of dice for groups or TI-graphing calculators (or any calculator that can simulate rolling a die.) (Activity 7.2.1)

PowerPoint presentation: *Tables* (Activity 7.2.1)

Rectangular box, paper for cutting shapes, scissors, rulers, compass, containers of small beans (or beads), paper cup, tape or glue stick. (Activity 7.2.2)