**Unit 8: Matrices**

**UNIT OVERVIEW**

13 days for investigations 1 – 4

24 days for investigations 1 – 6, testing and performance task

NOTE on use of this unit: This unit has been structured so that it will benefit students no matter how much of the unit can be addressed for a particular district.

Investigation 1 as a stand-alone: Students experience matrices as mathematical objects that have well defined operations. Students compare and contrast operations on matrices with operations on real numbers. They also see matrices as having real world utility.

Investigation 2 as a stand-alone: Students visit vectors and vector notation. There is much to be gained by spending time on vectors because too often students do not have facility with vectors and their uses. This investigation ultimately leads to students realizing that a vector can be thought of as a matrix.

Investigations 1 through 3: These three investigations ground students in matrix operations and introduces the notion of a matrix inverse. This is foundational for the utility of matrices as a method of solving large systems of equations. If there is only time for the first three investigations, it will be time well spent. It can lead to a much more mature view of mathematical objects and operations and the utility of vectors and matrices.

Investigation 1 through 4: Students who can complete the first four investigations will gain the additional benefit of understanding how to find the inverse of a 2X2 matrix (if it exists) and use the inverse to solve a matrix equation and find the solution to a system of equations. This is a major step in mathematical maturity; find a solution to a system of equations without graphing and without algebra.

The entire unit: Students who complete the entire unit will come to the realization that matrices are very useful objects and understanding matrix operations is a genuine introduction into abstract algebra as well the basic knowledge that these objects may or may not follow the rules that govern operations with real numbers.

Investigation 6 is not required for the unit, but it gives students with a strong background the opportunity to see matrices used to find solutions to interesting problems.

**The** **OVERVIEW**

This unit is a study of the application of matrices to appropriate contextual problems for the second year of high school algebra. Matrices are introduced as a natural extension for common methods of sorting and combining data in common life contexts. Once the basic form of matrices has been developed, students learn about the operations that apply to matrices and the mathematical properties that matrices follow. This provides a contrast to the real number system in the sense that some, but not all properties apply and some are violated. This provides an experience with mathematical structures that exhibit different behaviors with regard to sets and operations on sets. Matrices are examined in the context of a school project regarding reduce/reuse/recycle which is an initiative of our state DEEP. Reusing textile resources benefits the environment by reducing greenhouse gas emissions and providing funds to organizations that assist those in need.

Once a familiarity with matrices and operations with matrices is established, the next investigations gradually build on matrix algebra, the notion of the determinant, the inverse of a matrix and procedures for finding the inverse. This allows for solving matrix equations and solving systems of linear equations of various sizes. While students have encountered systems throughout algebra, solutions become more and more sophisticated and gradually move toward understanding the relationship between column spaces and row spaces of matrices and ultimately will provide the tool for understanding vector spaces and the basis for a vector space. This unit goes as far as solving 2×2 and 3×3 matrices by finding the inverse of a matrix of coefficients and using matrix algebra.

The final investigation ties many ideas together from the entire sequence of units in Algebra 2 and geometry by revisiting geometric ideas and matrices as being able to transform points, lines, and *n-*gons using a transformation matrix and that the inverse matrix reverses the result of the original transformation. This is a natural extension of the idea of inverse functions and shows that the fabric of mathematics is rich with ideas and notions that have analogs in other branches of mathematics.

**Investigation 1** This investigation introduces students to matrices as a compact tool for organizing contextual data and to operations that can be performed with matrices. Next, the idea of operations on matrices is examined and various operations with real numbers are seen to apply to matrices, while others are violated. Multiplication of matrices is introduced. Technology is explored as a way of working with matrices to speed the process of working with large quantities of data.

**Investigation 2** . This investigation starts with a review of vectors and the properties of vector quantities. Students learn that several notations can be used to define vectors and that representing a vector as an *m*×1 matrix is helpful for matrix operations, particularly for using transformation matrices to transform geometric objects.

**Investigation 3** Students experience some contextual frameworks for multiplying matrices and will experience the utility of multiplying matrices to solve problems. Additionally, they see that matrix multiplication can be understood in terms of the entries in the two matrices and that commutativity is violated because reversing the operation does not make contextual sense. They are again challenged to understand the inverse of a matrix.

**Investigation 4** Students become more familiar with the inverse of a matrix and learn how to find the inverse (if it exists), by using a matrix of cofactors and using cofactor expansion to find the determinant. This investigation is technically challenging for students and it culminates with using technology once students have demonstrated the ability to find inverse matrices “by hand”. The investigation includes rigorously solving a complex system of three equations using matrix algebra.

**Investigation 5** Students now see that a matrix can be a geometric transformation of points, lines and *n*-gons and that corresponding points are transformed by stretching and rotation based on the matrix. The inverse matrix undoes the effect of the original matrix. This ties in very closely to work on transformations in geometry and the compact representation of a transformation using a matrix. It foreshadows the use of matrices to represent change of bases for vector spaces.

**Investigation 6** In this investigation students explore the application of matrix algebra to stochastic processes. Students learn to compare and contrast processes in which subsequent events are independent of the result of a previous trial to processes in which subsequent events are dependent upon the preceding states of the variables. They explore the application of matrix algebra using state vectors and stochastic matrices and apply these mathematical constructs to contextual stochastic processes. They then apply matrix algebra to successive iterations to find a steady state if one exists.

**Essential Questions**

* What is a matrix as a mathematical object?
* What is the advantage of matrix representation of data from contextual problems?
* Which properties of the real number system apply to matrices and which do not?
* What operations can be performed upon matrices?
* How can a system of equations be represented and solved using matrix equations?
* How can geometric transformations be accomplished with matrix algebra?
* How can we use the inverse of a matrix (if it exists) to solve a matrix equation?

**Enduring Understandings**:

* Matrices can be used to represent data in an organized format.

**Unit Understandings**:

* There exist operations that are not commutative.
* Not all mathematical operations operate on numbers.
* Systems of linear equations can conveniently be solved using matrix algebra.
* Certain square matrices have inverses and these inverses can be determined using a matrix of cofactors.
* The inverse of a matrix can be used to help solve systems of linear equations that are expressed as a matrix equation.

**Unit Contents**

Investigation 1: Operations with Matrices (3 - 4 days)

Investigation 2: Operations with Vectors (4 days)

Investigation 3: Applications with Vectors and Matrices (3 days)

Investigation 4: Applications with *2* × *2* Matrices (3 - 4 days)

Investigation 5: Applications with *n* × *n* Matrices (4 days)

Investigation 6: Applications with Markov Chains and Stochastic Processes (4 days)

Performance Task: (1 day for presentations)

Review for Unit Test (1 day)

End of Unit Test (1 day)

**Common Core Standards for Mathematical Practice**

*Mathematical Practices #1 and #3* *describe a classroom environment that encourages thinking mathematically and are critical for quality teaching and learning. Practices in bold are to be emphasized in the unit.*

1. **Make sense of problems and persevere in solving them.**
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. **Look for and make use of structure.**
8. Look for and express regularity in repeated reasoning.

**Common Core State Standards**

N-VM 1Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., ***v***, |***v***|,||***v***||, ***v***).*,*

N-VM 2 Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N-VM 3 Solve problems involving velocity and other quantities that can be represented by vectors.

N-VM 4 Add and subtract vectors.

a. Add vectors end-to-end, component-wise, and by the

 parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

c. Understand vector subtraction ***v*** – ***w*** as ***v*** + (–***w***), where –***w*** is the additive inverse of ***w***, with the same magnitude as ***w*** and pointing in the opposite direction. Represent vector subtraction graphically

by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

N-VM 5 Multiply a vector by a scalar.

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as c(vx, vy) = (cvx, cvy).

b. Compute the magnitude of a scalar multiple cv using ||cv|| = |c|v. Compute the direction of cv knowing that when |c|v ≠ 0, the direction of cv is either along v (for c > 0) or against v (for c < 0).

N-VM 6 Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

N-VM 7 Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

N-VM 8 Add, subtract, and multiply matrices of appropriate dimensions.

N-VM 9 Understand that, unlike multiplication of numbers, matrix multiplication

 For square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

N-VM 10 Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers.

N-VM 11 Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

N-VM 12 Work with 2 X 2 matrices as a transformations of the plane, and interpret the absolute value of the determinant in terms of area.

**Assessment Strategies**

**Performance Task**

The Unit 8 Performance Task should be an open-ended assignment that requires students to solve a contextual problem with three variables using a matrix equation and finding the inverse of a 3×3 matrix. Ideally students would find the inverse by hand using a cofactor matrix and the determinant. However, technology may be permitted if students are at least required to exhibit familiarity with the steps in the procedure and how to identify an invertible matrix. Another option would be an assignment that frames a contextual probability as a stochastic process and asks students to determine the steady state.

**Other Evidence (Formative and Summative Assessments)**

* Exit slips
* Class work
* Homework assignments
* Math journals
* Unit 8 end of unit assessment

**Vocabulary**

A *matrix* is an rectangular array of numbers. A matrix is enclosed in brackets for example: $\left[\begin{matrix}&&\\&&\end{matrix}\right]$. The number of rows is listed first, then the number of columns.

So the example above is a 2×3 matrix.

A *greenhouse gas* is a gas that causes heat to be trapped in the atmosphere and contributes to global warming.

*Carbon dioxide* (CO2) is an example of a greenhouse gas. It is a product of combustion of carbon with oxygen.

The *magnitude* of a line segment is its length.

The *direction* of a line segment is its direction relative to some fixed direction.

A *vector* is a line segment that has both magnitude and direction so it is sometimes called a directed line segment.

A *scalar* is any real number.

The *determinant* is a value associated with a square matrix. It can be computed from the entries of the matrix by a specific arithmetic expression, while other ways to determine its value exist as well.

The *identity matrix* (represented as *I*) is a square matrix with the property that $AI=A$. It is a matrix with a 1 for each entry along the diagonal and all other entries are 0.

The inverse of a matrix *A* (represented as $A^{-1})$ is a matrix that has the property that $AA^{-1}=I$

A *coefficient matrix* is a matrix whose entries are the coefficients of the variables in a system of linear equations.

*Kirchhoff’s laws and Ohm’s law* are two laws governing electrical circuits that allow us to solve circuit using mathematical principles.

A *series circuit* is an electrical circuit where the electrical components are connected one after the other as they are in mini lights on a holiday tree.

A *parallel circuit* is an electrical circuit where the electrical components are connected in parallel like the rungs of a ladder.

A *transformation matrix* moves points in space according to the entries in the matrix. This transformation is reversible by using the inverse of the transformation matrix if it has an inverse.

A *Markov chain or Markov process* is a process in which the probability of the system being in a particular state at a given observation period depends only on its state at the immediately preceding observation period.

A *transition matrix* is a matrix *T* in which an entry $t\_{ij}$ is the probability of switching from state *j* to state *i* where there are *n* possible states. All entries must be nonnegative and all entries in a column must sum to one. Consequently a transition matrix is a probability matrix and is sometimes called a *stochastic matrix*.

The *state vector* of a Markov process with *n* states at time *step* ***k*** is the vector $\left[\begin{matrix}p\_{1}^{\left(k\right)}\\p\_{2}^{\left(k\right)}\\…\\p\_{n}^{\left(k\right)}\end{matrix}\right]$

where $p\_{j}^{\left(k\right)}$ is the probability of being in state *j* at step ***k*.**

A transition matrix (or corresponding Markov process) is called *regular* if some power of the matrix has all nonzero entries. Or, there is a positive probability of eventually moving from every state to every state.