**Unit 8: Investigation 5 (3 Days)**

**Fractals**

**Common Core State Standards**

* G-SRT-5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.
* F-BF-1 Write a function that describes a relationship between two quantities. (a) Determine an explicit expression, a recursive process, or steps for calculation from a context.
* F-BF-2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

**Overview**

In the launch discussion, students are introduced to fractals through a PowerPoint presentation that includes pictures of fractals in nature, the Julia Set, and fractal art. This discussion introduces students to the concept of self-similarity and to the idea that a complex image can be produced from repeating simple images at different scales. Fractal geometry holds some surprises. In Activity 1 starting with an equilateral triangle, students construct several stages of a Sierpinski triangle by hand. They establish the self-similar nature of the image at each stage and determine formulas for the perimeter and area. Students discover that as the stage number *n* increases, the perimeter increases but the area decreases. In fact, after a large number of stages, the area is nearly zero and the perimeter is huge. In Activity 2, students use GeoGebra to construct various stages of a dynamic Sierpinski triangle. In Activity 3, students use GeoGebra to construct various stages of a Koch Snowflake. They discover that the area of the Koch snowflake is finite, but the perimeter grows beyond bound as the stage number increases. In Activity 4 they investigate the Sierpinski carpet.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Be able to construct by hand several stages in a Sierpinski triangle.
* Determine a formula for calculating the perimeter and area for Stage *n* in the construction of a Sierpinski triangle.
* Be able to construct several stages in a Koch Snowflake.
* Determine a formula for calculating the perimeter for Stage *n* in the construction of a Sierpinski triangle and a Koch snowflake.
* Understand how to create and use a GeoGebra tool.

**Assessment Strategies: How Will They Show What They Know?**

* **Exit Slip 8.5** has studentsdetermine the area and perimeter of various stages of the square analog of the Koch snowflake.
* **Journal Entry** asks students to discuss how a fractal can have finite area but infinite perimeter.

**Launch Notes**

In this Investigation students explore a new type of geometry, fractal geometry. Begin this investigation with the PowerPoint: Fractals. Here are some pointers to use in the Launch discussion.

* Slides 2 and 3: The first image is of a fern. The second image shows the same fern with two sections outlined. The fern leaf has a particular shape that is similar (approximately) to the shape of the section of the fern that is outlined in red. If you could zoom in on the shape that is outlined in yellow, it would be similar to the section outlined in red, which is similar to the whole fern leaf. We say that the fern is approximately **self-similar**, which means that a small piece of the fern leaf is similar to the whole leaf. This is an example of a **fractal**, a pattern that repeats itself at different scales.
* Slides 4 and 5: The next image is a grape leaf. Focus on the veins in the leaf. There is a main vein down the center of the leaf, and off of this main vein there are major veins that run to the edge of the leaf. Off each of these major veins, there are other veins. You may be able to see this better in Slide 5 which shows the reverse side of a leaf. Ask students to describe the patterns that they see and compare them to the patterns on the fern.
* Slide 6: A lightning bolt is not a single line of light in the sky. It, too, has smaller repeating patterns.
* Slides 7, 8 and 9 show an image of food, a mathematical set called the Julia set, and fractal art. Give students an opportunity to discuss the self-similar nature of the patterns, and how patterns are repeated at different scales.

If you have some additional time, show a portion of the YouTube video: *Fractals – Hunting the Hidden Dimension*. This is a Nova episode that lasts 53 minutes. Students can watch the rest of this episode for homework. Here is the web address:

[www.youtube.com/watch?v=s65DSz78jW4](http://www.youtube.com/watch?v=s65DSz78jW4)

**Teaching Strategies**

 After completing the Launch discussion students are ready to begin **Activity 8.5.1**.

**Activity 8.5.1 Sierpinski Triangle**: Starting with an equilateral triangle, students construct Stages 1–3 of a Sierpinski triangle. At each stage, they calculate the area and perimeter. Then they predict what the area and perimeter will be at Stage *n*. They discover that as *n* gets large, the area approaches 0 but the perimeter grows without bound.

**Materials**: Students will need compasses (to construct the midpoints of the sides of an equilateral triangle) and rulers.

**Prerequisites**: Students should be able to:

* Construct the midpoint of a line segment using a compass (Unit 2 Investigation 7)
* Calculate the perimeter and area of an equilateral triangle given the length of one of its sides. (Unit 4 Investigation 8, Special Right Triangles)
* Calculate the area and perimeter of a triangle from the area and perimeter of a similar triangle with side lengths twice as long. (Unit 4 Investigation 2).

**Group Activity**Allow students to work in small groups for **Activity 8.5.1**. Students should construct several stages of a Sierpinski triangle individually. As part of a group, they can check to see if their constructions match with the constructions of other group members. In addition, students may have trouble coming up with formulas for the area and perimeter for the Stage *n* construction and it may take the efforts of the group to come up with a solution.

**Differentiated Instruction (For Learners Needing More Help)**For students who cannot determine a formula for the perimeter and area of Stage *n* in the construction of a Sierpinski triangle, walk them through the derivation or give them the formulas:

Perimeter of Stage *n*: $3∙\left(\frac{3}{2}\right)^{n}$

 Area of Stage *n*: $\frac{\sqrt{3}}{4}∙\left(\frac{3}{4}\right)^{n}$

To help students see the pattern of growth for the perimeter, have them graph,
*y* = $3∙\left(\frac{3}{2}\right)^{n}$where *x* represents the stage number *n*. Then have students graph
*y* =$ \frac{\sqrt{3}}{4}∙\left(\frac{3}{4}\right)^{n}$ For each function, ask: What happens as *n* (or *x*) gets really large?

**Activity 8.5.2 Sierpinski Triangle with GeoGebra**: The construction of stages of a Sierpinski triangle can be quite tedious. This activity provides instructions for using GeoGebra to construct Stages 0–5 of a dynamic Sierpinski triangle. In the construction, students learn how to create a GeoGebra tool that allows them to apply a process over and over again. In addition, they can change the shape of their Stage 0 triangle and see how later stages reflect this change.

**Activity 8.5.3 Koch Snowflake**: The Koch snowflake relates to coastline lengths. Students use a GeoGebra tool to construct Stages 0–3 of the Koch snowflake. At each stage, students determine the perimeter. Then they look for a pattern and write a formula for the perimeter at Stage *n*. Students find that the Koch snowflake can be surrounded by a circle and hence, its area must be smaller than that of the surrounding circle. However, as *n* increases, the perimeter grows beyond bound.

To give students a clearer view of Indiantown Harbor (shown in Figures 1 and 2 at the beginning of the activity), project the PowerPoint: Koch Snowflake. To make it easier to discuss the construction of the Koch snowflake, Stages 0–3 also appear in this PowerPoint.

**Differentiated Instruction (Enrichment)**

After students have completed Activity 8.6.3, ask them to find the area of Stages 1–4. Can they find a pattern? (Assume that the side length of the equilateral triangle in Stage 0 is 1 unit.)

Answer: Let *T* = the area of the Stage 0 triangle = $\frac{\sqrt{3}}{4}$ unit2. Then at each next stage *n* the new triangles each have area *T*$\left(\frac{1}{3}\right)^{2n}$= $\frac{T}{9^{n}}$ unit2. Three triangles are added at stage 1, 3$×4$ =12 triangles at stage 2, and in general, 3$×4^{n-1}$ triangles at stage *n.*

The total area of Stage *n* is thus *T*(1 + $\frac{3}{9}$ +$\frac{12}{81}$ +$\frac{48}{729}$ + … + $\frac{3×4^{n-1}}{9^{n}}$).

In a future course students may learn to sum an infinite geometric series to show that the area approaches $\frac{8T}{5}$ as *n* increases without bound.

**Activity 8.5.4 Sierpinski Carpet** introduces students to a fractal similar to the Sierpinksi triangle. As the stage number increases the area decreases but the perimeter increases. This characteristic is used in the production of antennas for mobile phones and WiFi systems.

**Journal Entry**
Earlier in this Geometry course, you have calculated perimeters and areas of various shapes, and in both cases, you determined a finite number. Explain how with fractals it is possible to have areas that are finite but perimeters that are infinite (in other words, grow without bound as the stage number increases). Look for students to explain that because amount added to the perimeter of the Koch snowflake increases each time, it will grow infinitely whereas the amount added to the area decreases each time, and the entire figure lies within a circle with finite area.

**Closure Notes**

Fractal geometry arises from creating patterns that are self-similar and repeat at different scales. They are useful mathematics to describe patterns in nature, such as coastlines, mountain ranges, leaf patterns, and lightning bolts. Point out that unlike what students have seen previously in this course, for fractals it is possible to have perimeters (or outlines of the fractal) that grow without bound (approach infinity) while areas remain finite. Review the specific fractals students have investigated: the Sierpinski triangle, the Sierpinski carpet, and the Koch snowflake.

**Vocabulary**

fractal

Koch snowflake

recursive formula

self-similar

Sierpinski carpet

Sierpinski triangle

**Additional Topics for Exploration**

For students conducting an independent study of fractals there are many additional topics they may want to explore. Here are several:

1. Fractal trees
2. The Mandelbrot set and Julia sets
3. Fractal dimension
4. Chaos and fractals
5. Fractals in nature

**Resources and Materials**

Activity 8.5.1 Sierpinski Triangle (Materials: Compasses, rulers)

Activity 8.5.2 Sierpinski Triangles Using GeoGebra

Activity 8.5.3 Koch Snowflake

Activity 8.5.4 Sierpinski Carpet

Exit Slip 8.6

PowerPoint presentation 8.6.1: Fractals (Launch)

PowerPoint presentation 8.6.2: Koch Snowflake

YouTube video for the NOVA episode (53 minutes) *Fractals – Hunting the Hidden Dimension*

[www.youtube.com/watch?v=s65DSz78jW4](http://www.youtube.com/watch?v=s65DSz78jW4)

YouTube video with instructions for making a Sierpinski triangle:

[www.youtube.com/watch?v=dJhtLVvtsj4](http://www.youtube.com/watch?v=dJhtLVvtsj4)

For more information on the Sierpinski triangle (or Sierpinski gasket), including an animated construction of the first 9 stages of a Sierpinski triangle, check out: en.wikipedia.org/wiki/Sierpinski\_triangle

For more information on the Koch snowflake, including an animated construction of the first 7 stages of the Koch Snowflake, check out: en.wikipedia.org/wiki/Koch\_snowflake

Print resources

Barnsley, Michael. *Fractals Everywhere*. Boston: Academic Press, 1988.

Briggs, John. *Fractals: The Patterns of Chaos*. New York: Simon and Schuster, 1992.

Devaney, R.; Choate, J., and Foster, A. *Fractals: A Tool Kit of Dynamics Activities,* Emeryville, CA: Key Curriculum Press, 1999.

Gleick, James. *Chaos: Making a New Science.* New York: Penguin Books, 1987.

Mandelbrot, Benoit. *The Fractal Geometry of Nature*. San Francisco: W. H. Freeman, 1983.